

# A GENERALIZATION OF INVERSION FORMULAS OF PESTOV AND UHLMANN

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ABSTRACT. In this note, we give a generalization of the inversion formulas of Pestov-Uhlmann for the geodesic ray transform of functions and vector fields on simple 2-dimensional manifolds of constant curvature. The inversion formulas given here hold for 2-dimensional simple manifolds whose curvatures close to a constant.

## 1. INTRODUCTION

Let  $(M, \partial M, g)$  be a  $C^\infty$  compact Riemannian manifolds with boundary. A variant of the classical Radon transform on Euclidean space is the *geodesic ray transform* on Riemannian manifolds defined as follows:

$$I_m f(\gamma) = \int_0^{l(\gamma)} \langle f(\gamma(t)), \dot{\gamma}^m(t) \rangle dt,$$

where  $\gamma : [0, l(\gamma)] \rightarrow M$  is a maximal geodesic parameterized by arc length and  $m$  indicates the rank of the symmetric tensor field  $f \in L^2(M)$ .

We will be interested only in the cases  $m = 0$  (functions) and  $m = 1$  (vector fields) in this note and denote their geodesic ray transforms by  $I_0$  and  $I_1$  respectively.

The geodesic ray transform is not injective in general. One needs additional restrictions on the metric and one such restriction is to assume that the Riemannian manifold  $(M, \partial M, g)$  is *simple* [Sha94] defined as follows:

**Definition 1.** *A compact Riemannian manifold with boundary is simple if*

- (a) *The boundary  $\partial M$  is strictly convex:  $\langle \nabla_\xi \nu, \xi \rangle < 0$  for  $\xi \in T_x(\partial M)$  where  $\nu$  is the unit inward normal to the boundary.*
- (b) *The map  $\exp_x : \exp_x^{-1} M \rightarrow M$  is a diffeomorphism for each  $x \in M$ .*

It is known that on a simple Riemannian manifold,  $I_0 f$  uniquely determines  $f$  and  $I_1 f$  uniquely determines the solenoidal component of  $f$ . For references to these works, we refer the book of Sharafutdinov [Sha94]. Then, similar to the classical Radon inversion formula, it is natural to ask whether there exists explicit inversion formulas for a function or a vector field in terms of its geodesic ray transform. In general this is a hard problem and such formulas are known in only in special cases [Hel99].

In [PU04], Pestov and Uhlmann found Fredholm-type inversion formulas for the geodesic ray transform of functions and vector fields for simple 2-dimensional manifolds. These formulas become exact inversion formulas for 2-dimensional manifolds of constant curvature, even when conjugate points are present along geodesics.

A brief remark regarding notation. We use notation that is standard in integral geometry literature. Ours is consistent for the most part with [Sha94]. In this note,  $SM$  is the unit sphere bundle and  $\tau(x, \xi)$  is the length of the maximal geodesic starting at  $x \in \partial M$  in the direction  $\xi \in \partial_+ SM := \{(x, \xi) \in \partial SM : \langle \nu(x), \xi \rangle \geq 0\}$ .

The Fredholm-type inversion formulas of Pestov-Uhlmann are given by the following theorem:

**Theorem 1.** [PU04, Theorem 5.4] *Let  $(M, g)$  be a 2-dimensional Riemannian manifold. Then*

$$f + \mathcal{W}^2 f = \frac{1}{4\pi} \delta_\perp I_1^* (\alpha^* H(I_0 f)^-|_{\partial_+ SM}), \quad f \in L^2(M).$$

$$h + (\mathcal{W}^*)^2 h = \frac{1}{4\pi} I_0^* (\alpha^* H(I_1 \mathcal{H}_\perp h)^+|_{\partial_+ SM}), \quad h \in H_0^1(M).$$

Here  $\mathcal{W}$  is the operator ( $\mathcal{W}^*$  is its  $L^2$  adjoint) on  $L^2(M)$  defined by

$$\mathcal{W}f(x) = \frac{1}{2\pi} \int_{S_x} \mathcal{H}_\perp \left( \int_0^{\tau(x, \xi)} f(\gamma_{x, \xi}(t)) dt \right) dS_x(\xi),$$

with

$$\mathcal{H}_\perp u(x, \xi) = \xi_\perp^i \left( \frac{\partial u}{\partial x^i} - \Gamma_{ij}^k \xi^j \frac{\partial u}{\partial \xi^k} \right).$$

As shown in [PU04], for manifolds of constant curvature,  $\mathcal{W} = \mathcal{W}^* = 0$  and hence these formulas becomes exact inversion formulas.

In this note, we generalize these formulas to simple 2-dimensional manifolds whose curvatures are close to a constant. We show that in this case, the inversion formulas are given by convergent Neumann series expansions. Our main result is a generalization of the above result:

**Theorem 2.** *There exists a  $C > 0$  such that if  $M$  is a simple 2-dimensional manifold with Gaussian curvature  $K$  such that  $\|\nabla K\|_{C^0} \leq C$ , the following inversion formulas hold:*

$$f = (I + \mathcal{W}^2)^{-1} \left( \frac{1}{4\pi} \delta_\perp I_1^* (\alpha^* H(I_0 f)^-|_{\partial_+ SM}) \right), \quad f \in L^2(M).$$

$$h = (I + (\mathcal{W}^*)^2)^{-1} \left( \frac{1}{4\pi} I_0^* (\alpha^* H(I_1 \mathcal{H}_\perp h)^+|_{\partial_+ SM}) \right), \quad h \in H_0^1(M).$$

Remark: The proof relies on getting bounds for the operator  $\mathcal{W}$  (and hence  $\mathcal{W}^*$ ) in terms of the gradient of the curvature  $K$ . Hence we will not give definitions of the terms appearing on the right hand side of the formulas in Theorems 1 or 2 which can be found in Pestov-Uhlmann's papers [PU05, PU04].

As shown in [PU04],  $\mathcal{W}$  is a smoothing integral operator extendible as a map  $\mathcal{W} : L^2(M) \rightarrow C^\infty(M)$  with kernel,

$$W(x, y) = -Q(x, \exp_x^{-1}(y)) \frac{|\det(\exp_x^{-1})'(x, y)| \sqrt{g(x)}}{\sqrt{g(y)}}. \quad (1)$$

The function  $Q$  (equations (6) and (7)) and the partial differential operator  $\partial_\theta$  (equation (4)) are defined in the appendix.

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## 2. THE PROOF

We prove the following lemma. Here the derivatives are with respect to time. The functions  $a$  and  $b$  are defined in the appendix; see equation (5).

**Lemma 1.** *Let  $\varphi = b\partial_\theta a - a\partial_\theta b$ . Then denoting  $K_\gamma = K \circ \gamma$ ,  $\varphi$  satisfies the following ordinary differential equation,*

$$\varphi^{(3)} + 4K_\gamma\varphi' + 2K'_\gamma\varphi = -2\partial_\theta K_\gamma,$$

with initial conditions,

$$\varphi(0) = \varphi'(0) = \varphi''(0) = 0.$$

*Proof.* First of all we have

$$ab' - a'b \equiv 1. \quad (2)$$

For, let  $\phi = ab' - ba'$ . Then  $\phi' = ab'' - a''b$ . From equation (5) we get that  $\phi' = 0$  and so  $\phi$  is a constant. Since  $\phi(0) = 1$ , we have the claim. With this we now show that  $\varphi = b\partial_\theta a - a\partial_\theta b$  satisfies the ODE above.

$$\varphi' = b'\partial_\theta a + b\partial_\theta a' - a'\partial_\theta b - a\partial_\theta b'.$$

From (2) we get

$$b'\partial_\theta a + a\partial b' - b\partial_\theta a' - a'\partial_\theta b = 0.$$

This gives

$$\varphi' = 2(b'\partial_\theta a - a'\partial_\theta b) = 2(b\partial_\theta a' - a\partial_\theta b').$$

Differentiating again, we get

$$\varphi'' = 2(b''\partial_\theta a + b'\partial_\theta a' - a''\partial_\theta b - a'\partial_\theta b').$$

Using equation (5), this reduces to

$$\varphi'' = 2(-K_\gamma\varphi + b'\partial_\theta a' - a'\partial_\theta b'),$$

where  $K_\gamma = K \circ \gamma$ . Differentiating yet again, and as in the steps above, we finally get,

$$\varphi^{(3)} + 2K'_\gamma\varphi + 4K_\gamma\varphi' = -2\partial_\theta K_\gamma, \quad (3)$$

It now follows directly from these equations that  $\varphi(0) = \varphi'(0) = \varphi''(0) = 0$ .  $\square$

Notation: The norm  $\|\cdot\|$  in the proof below denotes the sup norm unless indicated otherwise. Also in order to avoid proliferation of subscripts, we will use the same letter  $C$  to denote different constants.

*Proof of Theorem 2.* We now prove the main theorem. We rewrite equation (3) as a first order differential equation. We get

$$\begin{pmatrix} \varphi'_1 \\ \varphi'_2 \\ \varphi'_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2K'_\gamma & -4K_\gamma & 0 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -2\partial_\theta K_\gamma \end{pmatrix}$$

where

$$\varphi_1 = \varphi, \varphi_2 = \varphi', \text{ and } \varphi_3 = \varphi''.$$

For simplicity, let us write this as a system of the form

$$X'(t) = A(t)X(t) + B(t),$$

where  $X, B$  and matrix  $A$  depend also on  $(x, \xi)$ . From [Cod61], since  $X(0) = 0$ , we have a solution of this differential equation to be

$$X(t) = \Phi(t) \int_0^t \Phi^{-1}(s)B(s) ds.$$

where  $\Phi$  is the fundamental matrix of the homogeneous differential equation,

$$X'(t) = A(t)X(t).$$

Since the manifold is compact, we have  $\|\Phi\|, \|\Phi^{-1}\| < \infty$ . From the relation,

$$\partial_\theta K_\gamma = (\xi_\perp, \nabla K_\gamma),$$

and using the fact that  $SM$  is compact, we have a  $C$  such that

$$|\partial_\theta K_\gamma| \leq C\|\nabla K\|.$$

Combining these inequalities we get,

$$|\varphi'(x, \xi, t)| \leq |X(x, \xi, t)| \leq Ct\|\nabla K\|.$$

for some  $C > 0$ . Since

$$\varphi(t) = \int_0^t \varphi'(s) ds,$$

we have

$$|\varphi(t)| \leq Ct^2\|\nabla K\|.$$

We can initially work with  $\varphi''(t)$  and by the same arguments as above, we also get

$$|\varphi(t)| \leq Ct^3\|\nabla K\|,$$

for a different constant  $C$ .

Since the manifold is simple, we have  $b \neq 0$  for  $t \neq 0$ , since  $b(0) = 0$ . Now we write  $b(t, x, y) = tb(x, \tilde{y}, t)$  with  $\tilde{b} \neq 0$ . Therefore for a suitable  $C > 0$ ,

$$|q(x, \xi, t)| = |\partial_\theta \frac{a}{b}| \leq Ct\|\nabla K\|,$$

where the norm of  $\nabla K$  is the sup norm. Since  $tQ(x, t\xi) = q(x, \xi, t)$ , we have

$$|Q(x, t\xi)| \leq C\|\nabla K\|.$$

Since the remaining terms in

$$W(x, y) = -Q(x, \exp_x^{-1}(y)) \frac{\det(\exp_x^{-1})'(x, y)\sqrt{g(x)}}{\sqrt{g(y)}}$$

are bounded above by compactness of  $M$ , we have

$$\|W\| \leq C\|\nabla K\|.$$

Therefore we have

$$\|\mathcal{W}\|_{L^2 \rightarrow C^\infty(M)} \leq C\|\nabla K\|.$$

So now choosing  $\|\nabla K\|$  to be small enough, we have  $\|\mathcal{W}\| < 1$ . Hence we have inversion formulas involving Neumann series expansions recovering the function from its geodesic ray transform. A similar argument works for the recovery of the solenoidal part of a vector field from its geodesic ray transform. This completes the proof of the theorem.  $\square$

APPENDIX A. THE KERNEL OF  $\mathcal{W}$ 

For completeness and because the function  $q$  defined in equation (6) is critical for the proof of Theorem 2, we sketch below, Pestov-Uhlmann's [PU04] derivation of the integral kernel of the operator  $\mathcal{W}$ .

Recall that the operator  $\mathcal{W}$  is defined as

$$\mathcal{W}f(x) = \frac{1}{2\pi} \int_{S_x M} \mathcal{H}_\perp \int_0^{\tau(x, \xi)} f(\gamma(x, \xi, t)) dt dS_x,$$

where

$$\mathcal{H}_\perp = \xi_\perp^i \left( \frac{\partial}{\partial x^i} - \Gamma_{ij}^k \xi^j \frac{\partial}{\partial \xi^k} \right).$$

For a function  $u$  on  $SM$ ,  $\frac{\partial}{\partial \xi^k} u$  is defined as

$$\frac{\partial}{\partial \xi^k} u = \frac{\partial}{\partial \xi^k} (u \circ p)|_{|\xi|=1}, \text{ where } p(x, \xi) = (x, \xi/|\xi|).$$

We have

$$\mathcal{W}f(x) = \frac{1}{2\pi} \int_{S_x M} \int_0^{\tau(x, \xi)} \langle \nabla f, \mathcal{H}_\perp \gamma \rangle.$$

We define two Jacobi vector fields along the geodesic  $\gamma(x, \xi, t)$  as follows: Let  $x(s), -\varepsilon < s < \varepsilon$  be a curve starting at  $x$  in the direction  $\xi_\perp$ . Now parallel translate the vector  $\xi$  along this curve, call it  $\xi(s)$  and consider the variation by geodesics,  $\gamma(x(s), \xi(s), t)$ . The vector field

$$X(x, \xi, t) = \frac{d}{ds}|_{s=0} \gamma(x(s), \xi(s), t),$$

is a Jacobi vector field along  $\gamma$  with the following initial conditions,

$$X(x, \xi, 0) = \xi_\perp, \quad D_t X(x, \xi, 0) = 0.$$

It can be also be written as

$$X(x, \xi, t) = \mathcal{H}_\perp \gamma(x, \xi, t).$$

We now define another Jacobi vector field by considering the variation by geodesics,  $\gamma(x, \xi(s), t)$ , where  $\xi(s)$  is a smooth curve in  $S_x M$  with initial tangent vector  $\xi_\perp$ . The Jacobi vector field

$$\partial_\theta \gamma(x, \xi, t) := Y(x, \xi, t) = \frac{d}{ds}|_{s=0} \gamma(x, \xi(s), t) \quad (4)$$

has initial conditions,

$$Y(x, \xi, 0) = 0, \quad D_t Y(x, \xi, 0) = \xi_\perp.$$

Since  $X$  and  $Y$  are vector fields normal to  $\gamma$  and because of dimensional reasons these two fields must be proportional to the parallel translate of the vector  $\xi_\perp$  along the geodesic  $\gamma$ . Let this parallel translate be denoted  $\dot{\gamma}_\perp$ . Then there exists two smooth functions  $a(x, \xi, t)$  and  $b(x, \xi, t)$  such that

$$X = a \dot{\gamma}_\perp, \quad Y = b \dot{\gamma}_\perp.$$

The functions  $a$  and  $b$  satisfy the scalar Jacobi equations,

$$a'' + Ka = b'' + Kb = 0, \quad (5)$$

with

$$a(x, \xi, 0) = 1, \quad a'(x, \xi, 0) = 0, \quad b(x, \xi, 0) = 0, \quad b'(x, \xi, 0) = 1.$$

We now write,

$$\begin{aligned} \mathcal{W}f(x) &= \frac{1}{2\pi} \int_{S_x M} \int_0^{\tau(x, \xi)} \langle \nabla f, \mathcal{H}_\perp \gamma \rangle dt dS_x \\ &= \frac{1}{2\pi} \int_{S_x M} \int_0^{\tau(x, \xi)} \frac{a}{b} \langle \nabla f, Y \rangle dt dS_x \\ &= -\frac{1}{2\pi} \int_0^{\tau(x, \xi)} \int_{S_x M} \partial_\theta \left( \frac{a}{b} \right) f \circ \gamma dS_x dt. \end{aligned}$$

We now define a function  $q$  on

$$G = \{(x, \xi, t) : (x, \xi) \in SM, -\tau(x, -\xi) < t < \tau(x, \xi), t \neq 0\}$$

by

$$q(x, \xi, t) = \partial_\theta \left( \frac{a}{b} \right). \quad (6)$$

Using this we define a function  $Q \in C^\infty(TM)$  by,

$$Q(x, t\xi) = tq(x, \xi, t). \quad (7)$$

The existence of this function  $Q$  follows from the fact that the geodesic  $\gamma(x, \xi, t)$  is smooth as a function of  $(x, t\xi)$  and the initial conditions of the Lemma 1. We now get the integral kernel  $W(x, y)$  in equation (1) by a change of variables involving the inverse of the exponential map.

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